How many analytic sets are needed to form a MAD family?

Yhon J. Castro[*,1] and Osvaldo Guzmán[*,2]

[*]Corresponding authors E-mails: yhon@matmore.unam.mx, oguzman@matmore.unam.mx.

[1,2]Centro de Ciencias Matemáticas, UNAM, México.

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Abstract

The cardinal $\mathfrak{a}_{\mathbf{\Sigma}^1_1}$ is introduced. This is defined as the minimum quantity of analytic (*i,e*, $\mathbf{\Sigma}^1_1$) almost disjoint families needed in such a way that its union is a MAD family. This is a natural generalization of the cardinal $\mathfrak{a}_B$ introduced by J. Brendle and Y. Khomskii in [3]. In this paper, we present a proof, in **ZFC**, of the inequality $\mathfrak{h} \leq \mathfrak{a}_{\mathbf{\Sigma}^1_1}$, answering positively a conjecture made by D. Raghavan.

Keywords: Cardinal invariants, MAD families, Borel sets, Analytic sets.

1 Introduction

Given two sets $A, B \in [\omega]^\omega$, we say that they are *almost disjoint* (a.d.), if $A \neq B$ implies $|A \cap B| < \aleph_0$. An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is a.d. if all of its elements are a.d. Moreover, we say that $\mathcal{A}$ is a MAD *family* if it is a maximal a.d. family[1]. The study of definable subfamilies of $\mathcal{P}(\omega)$ has existed since the birth of set theory itself, and MAD families have been a fruitful object of study in this research line. It makes sense because identifying $\mathcal{P}(\omega)$ with 2^ω, we can talk about the definability of subfamilies in $\mathcal{P}(\omega)$ concerning Borel or projective hierarchies. The first result regarding the definability of MAD families is due to A.R.D Mathias. He proved, in [11], that there are no analytic (*i,e,* $\mathbf{\Sigma}^1_1$) MAD families. More recently, in [19], Törnquist gave a new proof of this theorem, and in an unpublished paper, C. Conely and B. Miller gave a shorter proof using the

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[1]The existence of such families follows easily from Zorn's Lemma. However, there are no MAD families in Solovay's model [16].

ideas from [19]. In contrast, A. Miller proved the existence of $\mathbf{\Sigma}^1_2$-definable MAD families in $\mathbf{L}$ [12]. A key part of his construction is that $\mathbf{L}$ has a $\mathbf{\Sigma}^1_1$-definable well order of the reals.

Closely related to questions of the definability of MAD families are problems concerning MAD families that can be expressed as the union of definable almost disjoint families. In [3], J. Brendle and Y. Khomskii constructed a MAD family $\mathcal{A}$ as the union of $\aleph_1$ Borel almost disjoint families $\{\mathcal{B}_\alpha\}_{\alpha<\aleph_1}$, using this construction to answer a question posed by S.D. Friedman and L. Zdomskyy in [5].

Specifically, Friedman and Zdomskyy asked whether it is consistent to have a $\mathbf{\Sigma}^1_2$ MAD family together with $\mathfrak{b} > \aleph_1$. Recall that increasing the bounding number $\mathfrak{b}$ requires adjoining dominating reals to the ground model. However, it is well known that adding a dominating real destroys all MAD families from the ground model. To avoid this, Brendle and Khomskii designed their family $\mathcal{A}$ so that the union of the reinterpretations of the Borel families $\{\mathcal{B}_\alpha\}_{\alpha<\aleph_1}$ remains maximal in the forcing extension that they used.

Motivated by this, they isolated the cardinal characteristic $\mathfrak{a}_B$, defined as follows:

Definition 1. *([3]) Let κ be an uncountable cardinal.*

1. *A MAD family $\mathcal{A} \subseteq [\omega]^\omega$ is called a κ-Borel MAD family if there are, for each $\alpha < \kappa$, Borel a.d. families $\mathcal{A}_\alpha$, such that $\mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$.*
2. *$\mathfrak{a}_B$ is the least cardinal κ for which there exists a κ-Borel MAD family.*

Inspired by this definition, it is natural to generalize to arbitrary pointclasses $\mathbf{\Gamma}$ (*e.g.* $\mathbf{\Gamma}$ is the collection of closed subsets, F_σ subsets, or analytic subset, etc).

Definition 2. *Let κ be an uncountable cardinal and $\mathbf{\Gamma}$ pointclass on $\mathcal{P}(\omega)$.*

1. *A MAD family $\mathcal{A} \subseteq [\omega]^\omega$ is called a κ-$\mathbf{\Gamma}$ MAD family if there are, for each $\alpha < \kappa$, a.d. families in the class $\mathbf{\Gamma}$, $\mathcal{A}_\alpha$, such that $\mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$.*
2. *$\mathfrak{a}_{\mathbf{\Gamma}}$ is the least cardinal κ for which there exists a κ-$\mathbf{\Gamma}$ MAD family.*

Theorem 3 (Main theorem)**.** $\mathfrak{h} \leq \mathfrak{a}_{\mathbf{\Sigma}^1_1}$.

D. Raghavan conjectured that $\mathfrak{h} \leq \mathfrak{a}_B$, and since $\mathfrak{a}_{\mathbf{\Sigma}^1_1} \leq \mathfrak{a}_B$, the previous theorem confirms it. Recall that the *distributivity number*, the cardinal $\mathfrak{h}$, is defined as follows:

Definition 4. 1. *Given a family $D \subseteq [\omega]^\omega$ we say that D is:*

Open: If for every $B \in D$ and $A \in [\omega]^\omega$, we have that $A \subseteq^* B$[2], implies $A \in D$.

Dense: If for every $B \in [\omega]^\omega$, there is $A \in D$ such that $A \subseteq B$.

[2]Recall that $A \subseteq^* B$ means that $A \setminus B$ is finite.

2. *The cardinal $\mathfrak{h}$ is defined as the minimum size of a family of open and dense sets in $[\omega]^\omega$ with an empty intersection.*

The proof of Theorem 3 relies on two central tools: The first one is the ideas of D. Schrittesser and A. Törnquist, to give a positive answer to a long-standing unsolved problem of Mathias. To be precise, to prove that, under **ZF**+**DC**+**R-Unif**+*"all sets have the Ramsey property"*, there are no infinite MAD families [16]. The second one is the characterization of the cardinal $\mathfrak{h}$ established by S. Plewik in [13] (see also [7]).

Theorem 5 (S. Plewik)**.** $\mathfrak{h} = cov(\mathcal{R}_0)$[3].

Moreover, as D. Raghavan has pointed out to us (private communication), our result implies that under $\mathbf{PFA}(S)[S]$, $\mathfrak{a}_{\mathbf{\Sigma}^1_1} = \aleph_2$. This is because he and T. Yorioka showed in [15] that under this hypothesis $\mathfrak{h} = \aleph_2$.

Before we give a proof of Theorem 3, we must introduce some necessary tools. In the second part of the paper, we will develop those tools, leaving the proof of the main theorem for the last section. Finally, we will point out some results that are already known about the cardinals $\mathfrak{a}_{\mathbf{\Gamma}}$, more precisely, when $\mathbf{\Gamma}$ is the class of Borel or closed sets.

Theorem 6.

1. *(Mathias) $\aleph_1 \leq \mathfrak{a}_B \leq \mathfrak{a}$ (see [11]).*
2. *(Raghavan and Shelah) $\mathfrak{d} = \aleph_1 \to \mathfrak{a}_{closed} = \aleph_1$ (see [14]).*
3. *(Raghavan) $\mathfrak{t} \leq \mathfrak{a}_B$ (unpublished).*
4. *(Brendle and Khomskii) It is consistent that $\mathfrak{a}_B < \mathfrak{b}$ (see [3]).*
5. *(Brendle and Raghavan) $\aleph_1 = \mathfrak{b} < \mathfrak{a}_{closed} = \aleph_2$ is consistent (see [2]).*
6. *(Törnquist) $\mathfrak{p} \leq \mathfrak{a}_B$ (see [19]).*[4]
7. *(Guzmán and Kalajdzievski) It is consistent that $\aleph_1 = \mathfrak{u} < \mathfrak{a}_{closed}$ (see [6]).*

The definitions of the almost disjoint number $\mathfrak{a}$, dominating number $\mathfrak{d}$, bounding number $\mathfrak{b}$, tower number $\mathfrak{t}$, pseudointersection number $\mathfrak{p}$ and ultrafilter number $\mathfrak{u}$, can be found in [1].

2 Preliminaries

In this section, we will introduce all of the necessary notions to follow the proof of our main theorem. We will do our best to keep the paper as self-contained as possible. Our notation is mostly standard, and the reader can consult [1, 9, 7] for a deeper development of the concepts used in the paper.

[3]See the paragraph after Definition 7 for the definition of $\mathsf{cov}(\mathcal{R}_0)$.

[4]By the celebrated result of Malliaris-Shelah [10], this result is the same as Raghavan's unpublished one. However, Raghavan's result was shown before it was known that $\mathfrak{p} = \mathfrak{t}$.

Let $A \in [\omega]^\omega$ and $s \in [\omega]^{<\omega}$. By $s \sqsubseteq A$ we mean that s is an *initial segment* of A, that is, $s \subseteq A \subseteq s \cup A \setminus \bar{s}$ where $\bar{s} := \sup\{k+1 : k \in s\}$. $[s, A]^\omega$ stands for the family $\{X \in [\omega]^\omega : s \subseteq X \subseteq s \cup A \setminus \bar{s}\}$. It is easy to see that the sets $[s, A]^\omega$ form a base for a topology, and that topology is known as the *Ellentuck's topology.* In particular, note that $\langle s \rangle := [s, \omega]^\omega$ is a base for the classical Polish topology of $[\omega]^\omega$. The following definition is due to J. Silver [17].

Definition 7. *A family $C \subseteq [\omega]^\omega$ is completely Ramsey if for every $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, there is $B \in [A]^\omega$ such that $C \cap [s, B]^\omega = \emptyset$ or $[s, B]^\omega \subseteq C$.*

If in the previous definition, for every $A \in [\omega]^\omega$ and $s \in \omega^{<\omega}$, we always find $B \in [A]^\omega$ such that $C \cap [s, B]^\omega = \emptyset$, we will say that C is a *completely Ramsey-null* set. It is easy to see that the set $\mathcal{R}_0 := \{C \subseteq [\omega]^\omega : C \text{ is completely Ramsey-null}\}$ forms an ideal in $\mathcal{P}([\omega]^\omega)$. That is, it contains the finite families of infinite sets, it is closed under taking subsets and finite unions, and does not contain $[\omega]^\omega$. Moreover, it is a *σ-ideal*, i.e., it is closed under countable unions. Recall that $\mathsf{cov}(\mathcal{R}_0)$ denotes the minimum cardinal κ for which there are κ-many elements in $\mathcal{R}_0$ whose union covers $[\omega]^\omega$.

The principal feature of Ellentuck's topology is that it nicely characterizes combinatorial notions like being a completely Ramsey set or a completely Ramsey-null set in topological terms.

Theorem 8 (Ellentuck's Theorem [4, 18])**.** *For every $A \subseteq [\omega]^\omega$ we have:*

1. *A is nowhere dense with respect to the Ellentuck topology if, and only if, A is completely Ramsey-null.*
2. *A has the Baire property with respect to the Ellentuck topology if, and only if A is completely Ramsey.*

Recall that in a topological space (X, τ), $Y \subseteq X$ has the *Baire property* if there is an open set $G \in \tau$ such that $Y \triangle G$ is meager[5]. Since the Baire property is closed under Souslin operation, and every classical closed set is also closed in Ellentuck's topology, classical analytic sets have the Baire property in Ellentuck's topology and, therefore, classical analytic sets are completely Ramsey sets.

The following result is a kind of fusion lemma for the family of completely Ramsey sets.

Lemma 9. *Let $\{R_n\}_{n\in\omega}$ be a countable family of completely Ramsey sets, $M \in [\omega]^\omega$ and $t \in [\omega]^{<\omega}$. Then:*

1. *There exists a set $N \in [M \setminus \bar{t}]^\omega$ with increasing enumeration $\{n_k : k \in \omega\}$ such that, for every $k \in \omega$ and $s \subseteq \{n_0, \ldots, n_k\}$ we have that $[t \cup s, N]^\omega \subseteq R_i$ or $[t \cup s, N]^\omega \cap R_i = \emptyset$ for every $i \leq \max(s)$.*

[5]Recall that $\triangle$ denotes the *symmetric difference.*

2. *Assume that for each $n \in \omega$, R_n is closed under $=^*$. Then there exists a set $N \in [M \setminus \bar{t}]^\omega$ with increasing enumeration $\{n_k : k \in \omega\}$ such that, for every $k \in \omega$ and $s \subseteq \{n_0, \dots, n_k\}$ we have that $[t \cup s, N]^\omega \subseteq R_i$ or $[t \cup s, N]^\omega \cap R_i = \emptyset$ for every $i \leq k$.*

3. *Let N the set given by 2. and $i < \omega$. Then $[t, N]^\omega \subseteq R_i$ if and only if there exists $s \in [N]^{<\omega}$ such that $[t \cup s, N]^\omega \subseteq R_i$. In particular, for every $i \in \omega$, either $[t, N]^\omega \subseteq R_i$ or $[t, N]^\omega \cap R_i = \emptyset$.*

Proof. 1. We are going to build the set N recursively. Let $n_0 := \min M \setminus \bar{t}$. Since for each $i \leq n_0$, the sets R_i are completely Ramsey, we can choose a decreasing sequence $\{N_0^i\}_{i \leq n_0}$ of elements in $[M \setminus \bar{t}]^\omega$ such that $[t \cup \{n_0\}, N_0^i] \subseteq R_i$ or $[t \cup \{n_0\}, N_0^i] \cap R_i = \emptyset$. Let $N_0 := N_0^{n_0}$. It is clear that for all $i \leq n_0$, $[t \cup \{n_0\}, N_0] \subseteq R_i$ or $[t \cup \{n_0\}, N_0] \cap R_i = \emptyset$.

Assume that we already have a decreasing sequence of infinite sets $N_k \subseteq N_{k-1} \subseteq \cdots \subseteq N_0$ and an increasing sequence $n_0 < n_1 < \cdots < n_k$ such that for every $s \in \mathcal{P}(\{n_0, \dots, n_k\})$, if $\max(s) = n_j$, then for every $i \leq n_j$ we have that $[t \cup s, N_j]^\omega \subseteq R_i$ or $[t \cup s, N_j]^\omega \cap R_i = \emptyset$. Let $n_{k+1} = \min(N_k \setminus (n_k + 1))$. Let $\{s_j : j \in 2^{k+1}\}$ be an enumeration of $\mathcal{P}(\{n_0, \dots, n_k\})$. Since for every $i \leq n_{k+1}$ the set R_i is completely Ramsey, we can construct a decreasing sequence

$$N_k^{(2^{k+1}-1, n_{k+1})} \subseteq \cdots \subseteq N_k^{(2^{k+1}-1, 0)} \subseteq N_k^{(2^{k+1}-2, n_{k+1})} \cdots \subseteq N_k^{(0,0)} \subseteq N_k$$

such that for every $j \in 2^{k+1}$ and $i \leq n_{k+1}$ we have that $[t \cup s_j \cup \{n_{k+1}\}, N_k^{(j,i)}]^\omega \subseteq R_i$ or $[t \cup s_j \cup \{n_{k+1}\}, N_k^{(j,i)}]^\omega \cap R_i = \emptyset$. Let $N_{k+1} := N_k^{(2^{k+1}-1, n_{k+1})}$. Pick $s \in \mathcal{P}(\{n_0, \dots, n_k\})$ and $i \leq n_{k+1}$. Take $j \in 2^{k+1}$ such that $s = s_j$. Then, as $[t \cup s \cup \{n_{k+1}\}, N_{k+1}]^\omega \subseteq [t \cup s_j \cup \{n_{k+1}\}, N_k^{(j,i)}]^\omega$ we obtain that $[t \cup s \cup \{n_{k+1}\}, N_{k+1}]^\omega \subseteq R_i$ or $[t \cup s \cup \{n_{k+1}\}, N_{k+1}]^\omega \cap R_i = \emptyset$. Finally, take $N := \{n_k : k \in \omega\}$. We claim that this N satisfies the requirements. Indeed, let $k \in \omega$, $s \in \mathcal{P}(\{n_0, \dots, n_k\})$ and $n_j := \max(s)$. By the way in which N was constructed, we get $[t \cup s, N]^\omega \subseteq [t \cup s, N_j]^\omega$. So, for every $i \leq n_j$, we conclude that $[t \cup s, N]^\omega \cap R_i = \emptyset$ or $[t \cup s, N]^\omega \subseteq R_i$.

2. Let N the set given by 1. and take $k \in \omega$, $s \in \mathcal{P}(n_0, \dots, n_k)$ and $i \leq k$. If $i \leq \max(s)$, the conclusion follows by the definition of N. Assume that $\max(s) < i$. We know that $[t \cup s, N_i]^\omega \subseteq R_i$ or $[t \cup s, N_i] \cap R_i = \emptyset$. Assume that the first condition happens. Let $X \in [t \cup s, N]^\omega$ and define $Y := X \setminus \{n_r : \max(s) < r \leq i\}$. So, $Y \in [t \cup s, N_i]^\omega \subseteq R_i$. But, since R_i is closed under $=^*$, and $X =^* Y$, we get $X \in R_i$. So, $[t \cup s, N]^\omega \subseteq R_i$. On the other hand, if $[t \cup s, N_i]^\omega \cap R_i = \emptyset$ happens, with the same argument as before, we can conclude that $[t \cap s, N]^\omega \cap R_i = \emptyset$.

3. The proof of $(\rightarrow)$ is trivial. Now, for $(\leftarrow)$, assume that $[t \cup s, N]^\omega \subseteq R_i$. Since R_i is closed under $=^*$, it is easy to conclude that $[t, N]^\omega \subseteq R_i$

$\square$

Definition 10. *([16]) Let $\{A_n\}_{n\in\omega}$ be a partition of ω in infinite sets. For any $Y \in [\omega]^\omega$ we define*

$$\hat{Y} := \{a_{y_n}(y_{n+1}) : n \in \omega\}$$

Where $\{y_n : n \in \omega\}$ and $\{a_k(i) : i \in \omega\}$ are the increasing enumerations of Y and A_k respectively.

Notice that $|\hat{Y} \cap A_n| \leq 1$ for every $n \in \omega$. Moreover, if $X, Y \in [\omega]^\omega$ and $X =^* Y$, then $\hat{X} =^* \hat{Y}$. Also note that the definition of $\hat{Y}$ depends of the chosen partition $\{A_n\}_{n\in\omega}$. However, since we will only treat one partition at a time, we decide to drop the mention of it in the notation. Moreover, observe that we can also define $\hat{s}$ for every $s \in [\omega]^{<\omega}$.

Lemma 11. *Let $\{A_n\}_{n\in\omega}$ be a partition of ω in infinite sets. The function $g : [\omega]^\omega \to [\omega]^\omega$ given by $g(Y) := \hat{Y}$ is continuous.*

Proof. Let $Y \in [\omega]^\omega$ and $t \sqsubseteq \hat{Y}$. So, we can find $s \sqsubseteq Y$ such that $s = \{y_0, \ldots, y_n\}$ and $t = \{a_{y_0}(y_1), \ldots, a_{y_n}(y_{n+1})\}$. It is clear that $\hat{s} = t$, $g^{-1}(\langle \hat{s} \rangle) \supseteq \langle s \rangle$ and $Y \in \langle s \rangle$. So, continuity follows. $\square$

A partially ordered set $(T, \leq)$ is called a *tree* if for every $t \in T$ the set $\{s \in T : s \leq t\}$ is well-ordered. A maximal totally-ordered subset of T will be called a *branch*, and the maximal elements of T will be called *leaves*. A *subtree* of T is a set $S \subseteq T$ that is downwards-closed under $\leq$. Given $s \in T$, we can define the subtree $T_s := \{t \in T : t \leq s \vee s \leq t\}$. Clearly $\omega^{<\omega}$ (the set of finite sequences of natural numbers) endowed with the order $s \leq t$ if, and only if, $s \subseteq t$ is a tree. Now, if $T \subseteq \omega^{<\omega}$ is a subtree, by $[T]$ we denote the set of its branches. It is well-known that $F \subseteq \omega^\omega$ is closed if, and only if, $F = [T]$ for some subtree $T \subseteq \omega^{<\omega}$. Finally, by a classical result from descriptive set theory [8, C. II] we have:

Theorem 12. *Let X be a Polish space (i.e., completely metrizable and separable) and $A \subseteq X$ an analytic set. Then there is a continuous and onto function $f : \omega^\omega \to A$.*

We finalize this section with a result due to Törnquist [19], which plays an important role in proving Theorem 3; we prove it here for the reader's convenience.

Lemma 13. *Let $T \subseteq \omega^{<\omega}$ be a tree and $f : [T] \to [\omega]^\omega$ continuous such that $f([T])$ is an almost disjoint family. If T is such that $|f([T])| \geq 2$, then there exists $n \in \omega$ and $s, t \in T$ such that*

$$\forall A \in f([T_s]) \forall B \in f([T_t])(A \cap B \subseteq n).$$

Proof. Let $A, B \in f([T])$ such that $A \neq B$. Take $k \in \omega$ such that $A \cap k \neq B \cap k$. By the continuity of f, we can find $a, b \in [T]$ and $i \in \omega$ such that $f(a) = A, f(b) = B, \forall A' \in f(\langle a \restriction i \rangle)(A \cap k = A' \cap k)$ and $\forall B' \in f(\langle b \restriction i \rangle)(B \cap k = B' \cap k)$.

Assume that the Lemma is not true. Fix $n \in \omega$ and assume that for every extension $s, t \in T$ of $a{\restriction}i$ and $b{\restriction}i$, respectively, there exists $A' \in f([T_s])$ and $B' \in [T_t]$ such that $A' \cap B' \nsubseteq n$. So, we can find $a', b' \in [T]$ such that $s \subseteq a', t \subseteq b', f(a') = A'$ and $f(b') = B'$. Using the continuity of f again, we can get k sufficiently large such that $\forall X \in f([T_{a'{\restriction}k}])\forall Y \in f([T_{b'{\restriction}k}])(X \cap Y \nsubseteq n)$.

Using inductively the argument in the paragraph above, we find increasing sequences $\{s_n\}_{n\in\omega}$,$\{t_n\}_{n\in\omega} \subseteq T$ of extensions of $a{\restriction}i$ and $b{\restriction}i$ such that $\forall X \in f([T_{s_n}])\forall Y \in f([T_{t_n}])(X \cap Y \nsubseteq n)$. But note that $f(\bigcup_{n\in\omega} s_n)$ and $f(\bigcup_{n\in\omega} t_n)$ are neither equal nor almost disjoint. We have arrived at the desired contradiction. □

3 Proof of the main theorem

This section is exclusively devoted to the proof of our main result, *i,e.* $\mathfrak{h} \leq \mathfrak{a}_{\boldsymbol{\Sigma}^1_1}$.

Proof. (Theorem 3) Let $\kappa < \mathfrak{h}$ and $\{\mathcal{A}_\alpha\}_{\alpha<\kappa}$ a collection of analytic almost disjoint families such that $\mathcal{A} := \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$ is an almost disjoint family. We will show that $\mathcal{A}$ is not a MAD family. Without loss of generality we can assume that $\mathcal{A}_n := \{A_n\}$ for every $n \in \omega$ where $\{A_n : n \in \omega\}$ is a partition of ω in infinite sets. For every $\alpha \geq \omega$, by Theorem 12, we can find a continuous and onto function $f_\alpha : \omega^\omega \to \mathcal{A}_\alpha$.

Fix $\alpha \geq \omega$, take $T := \omega^{<\omega}$, $f := f_\alpha$, and define the set

$$N_\alpha := \{M \in [\omega]^\omega : \exists A \in \mathcal{A}_\alpha(|A \cap \hat{M}| = \aleph_0)\}$$

where $\hat{M}$ is defined like in Definition 10 using the partition $\{A_n : n \in \omega\}$.

Lemma 14. *The set N_α is analytic.*

Proof. Let $h : [\omega]^\omega \times \mathcal{A}_\alpha \to \mathcal{P}(\omega)$ such that $h(M, A) := \hat{M} \cap A$. By Lemma 11, h is continuous and this implies that $h^{-1}([\omega]^\omega)$ is a Borel set. Note that N_α is the projection in the first coordinate of $h^{-1}([\omega]^\omega)$. So, N_α is analytic. □

Using the previous lemma and Ellentuck's theorem, we conclude that N_α is completely Ramsey. Moreover, we have the following:

Claim 15. *N_α is a completely Ramsey-null set.*

Proof. Assume, towards a contradiction, that there is $M \in [\omega]^\omega$ and $t \in [\omega]^{<\omega}$ such that $[t, M]^\omega \subseteq N_\alpha$. For every $X \in [t, M]^\omega$ we define the set

$$T(X) := \{l \in T : \exists A \in f([T_l])(|\hat{X} \cap A| = \aleph_0)\}.$$

$T(X)$ is a non-empty tree without leaves (*i.e.*, is well-founded).

Subclaim 16. *There exists $N \in [M \setminus \bar{t}]^\omega$ and $S \subseteq T$ subtree such that, if $P \in [t, N]^\omega$, then $T(P) = S$.*

Proof. Given $l \in T$, we consider the set

$$W(l) := \{X \in [t, M]^\omega : l \in T(X)\}.$$

Again, it is easy to show that $W(l)$ is analytic, hence, $W(l)$ is completely Ramsey for every $l \in T$. Moreover, let $X \in W(l)$ and $Y \in [t, M]^\omega$ with $X =^* Y$. So, $\hat{X} =^* \hat{Y}$ which implies that $W(l)$ is closed under $=^*$. Let $T := \{l_n : n \in \omega\}$ be an enumeration of T. By Lemma 9, we can choose $N \in [M \setminus \bar{t}]^\omega$ such that for every $i \in \omega$, $[t, N]^\omega \subseteq W(l_i)$ or $[t, N]^\omega \cap W(l_i) = \emptyset$.

Let $S := \{l \in T : [t, N]^\omega \subseteq W(l)\}$. S is a subtree of T, because if $l, s \in T$ and $l \subseteq s$, then $W(s) \subseteq W(l)$. We claim that N and S are the required sets. Indeed, if $P \in [t, N]^\omega$, then:

1. $T(P) \subseteq S$: Let $l \in T(P)$. Hence $P \in W(l)$ which implies that $[t, N]^\omega \cap W(l) \neq \emptyset$. By how N was chosen, we have that $[t, N]^\omega \subseteq W(l)$, and this implies $l \in S$.

2. $S \subseteq T(P)$: If $l \in S$, $[t, N]^\omega \subseteq W(l)$ by definition of S. Hence $P \in W(l)$, thus $l \in T(P)$.

□

Let N and S be as in the previous sub-claim. Since S has the form $T(P)$ for every $P \in [t, N]^\omega$, we conclude that S does not have leaves and $T(P) = \{l \in T : \exists A \in f([S_l])(|\hat{X} \cap A| = \aleph_0)\}$. Additionally, we have the following:

Subclaim 17. $|f([S])| = 1$.

Proof. To get a contradiction, assume that $|f([S])| \geq 2$. By Lemma 13 we can find $m \in \omega$ and $s, l \in S$ such that

$$\forall A \in f([S_s]) \forall B \in f([S_l])(A \cap B \subseteq m).$$

Let $W := (\bigcup f([S_s])) \setminus m$ and consider the 2-coloring $c : [N]^2 \to 2$ defined by:

$$c(\{i, j\}) := \begin{cases} 1 & \text{if } a_i(j) \in W \quad (i < j) \\ 0 & \text{otherwise} \end{cases}$$

By the Ramsey's Theorem we can find $P \in [N]^\omega$ such that $c \restriction [P]^2$ is monochromatic. We have two cases:

Case 1: If P is 1-monochromatic, we claim that $\hat{P^t} \subseteq^* W$ where $P^t := t \cup P$. If this was not true and $\{p_n : n \in \omega\}$ is the increasing enumeration of P, we get $a_{p_n}(p_{n+1}) \notin W$ for some $n \in \omega$. But this is a contradiction because $c(\{p_n, p_{n+1}\}) = 1$. Now, given an arbitrary $B \in f([S_l])$, we have $\hat{P^t} \cap B \subseteq^* W \cap B = \emptyset$ which is a contradiction because $l \in S = T(P^t)$. So, this case is impossible.

Case 2: If P is 0-monochromatic, we can conclude that $\hat{P^t} \cap W =^* \emptyset$ using an argument like the previous one. This contradicts that $s \in S = T(P)$.

From this, we conclude that $|f([S])| = 1$. $\square$

Let $A \in f([T])$ such that $f([S]) = \{A\}$. Since $A \cap A_n$ is finite for all $n \in \omega$, we can find $K \in [N]^\omega$ such that $|\hat{K^t} \cap A| \leq |t|$. Moreover, $K^t \in [t, N]^\omega \subseteq N_\alpha$, so we can find $B \in f([T])$ such that $|B \cap \hat{K^t}| = \aleph_0$. But note that $T(K^t) = S$ so, we must have that $B \in f([S]) = \{A\}$ which is a contradiction. This allows us to get that N_α must be a completely Ramsey-null set. $\square$

Now, as $\kappa < \mathfrak{h} = cov(\mathcal{R}_0)$, we can find $X \in [\omega]^\omega \setminus (\bigcup_{\omega \leq \alpha < \kappa} N_\alpha)$. So, for every $\omega \leq \alpha < \kappa$ and $A \in f_\alpha[\omega^\omega]$, we have that $|A \cap \hat{X}| < \aleph_0$. Moreover, $|\hat{X} \cap A_n| \leq 1$ for every $n \in \omega$, so $\mathcal{A} \cup \{\hat{X}\}$ is an almost disjoint family, *i,e.* $\mathcal{A}$ is not maximal. $\square$

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References

[1] Blass, A. (2009). *"Combinatorial cardinal characteristics of the continuum"*. In Handbook of set theory (pp. 395-489). Dordrecht: Springer Netherlands.

[2] Brendle, J., & Raghavan, D. (2014). *"Bounding, splitting, and almost disjointness"*. Annals of Pure and Applied Logic, 165(2), 631-651.

[3] Brendle, J., & Khomskii, Y. (2013). *"Mad families constructed from perfect almost disjoint families"*. The Journal of Symbolic Logic, 78(4), 1164-1180.

[4] Ellentuck, E. (1974). *"A new proof that analytic sets are Ramsey"*. The Journal of Symbolic Logic, 39(1), 163-165.

[5] Friedman, S. D., & Zdomskyy, L. (2010). *"Projective mad families"*. Annals of Pure and Applied Logic, 161(12), 1581-1587.

[6] Guzmán, O., & Kalajdzievski, D. (2021). *"The ultrafilter and almost disjointness numbers"*. Advances in Mathematics, 386, 107805.

[7] Halbeisen, L. J. (2012). *"Combinatorial set theory (Vol. 121). London: Springer"*.

[8] Kechris, A. (2012). *"Classical descriptive set theory"* (Vol. 156). Springer Science & Business Media.

[9] Kunen, K. (2014). *"Set theory an introduction to independence proofs"*. Elsevier.

[10] Malliaris, M., & Shelah, S. (2016). *"Cofinality spectrum theorems in model theory, set theory, and general topology"*. Journal of the American Mathematical Society, 29(1), 237-297.

[11] Mathias, A. R. (1977). *"Happy families"*. Annals of Mathematical Logic, 12(1), 59-111.

[12] Miller, A. W. (1989). *"Infinite combinatorics and definability"*. Annals of Pure and Applied Logic, 41(2), 179-203.

[13] Plewik, S. (1986). *"On completely Ramsey sets"*. Fund. Math, 127, 127-132.

[14] Raghavan, D., & Shelah, S. (2012). *"Comparing the closed almost disjointness and dominating numbers"*. Fund. Math., 217(1), 73–81.

[15] Raghavan, D., & Yorioka, T. (2012). *"Some results in the extension with a coherent Suslin tree"*. Aspects of Descriptive Set Theory, Sūrikaisekikenkyūsho Kōkyūroku, (1790):72–82.

[16] Schrittesser, D., & Törnquist, A. (2019). *"The Ramsey property implies no mad families"*. Proceedings of the National Academy of Sciences, 116(38), 18883-18887.

[17] Silver, J. (1970). *"Every analytic set is Ramsey"*. The Journal of Symbolic Logic, 35(1), 60-64.

[18] Todorcevic, S. (2010). *"Introduction to Ramsey spaces"* (am-174) (Vol. 174). Princeton University Press.

[19] Törnquist, A. (2018). *"Definability and almost disjoint families"*. Advances in Mathematics, 330, 61-73.